

Reductions of Topologically Massive Gravity I: Hamiltonian Analysis of The Second Order Degenerate Lagrangians

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Abstract: We study the Hamiltonian formalisms of the second order degenerate Clément and Sarioğlu-Tekin Lagrangians. The Dirac-Bergmann constraint algorithm is employed while arriving at the total Hamiltonian functions and the Hamilton's equations on the associated momentum phase spaces whereas the Gotay-Nester-Hinds algorithm is run while investigating the Skinner-Rusk unified formalism on the proper Whitney bundles.

Key words: Second order degenerate Lagrangians, Dirac-Bergmann algorithm, Sarioğlu-Tekin Lagrangian, Clément Lagrangian, Skinner-Rusk unified formalism.

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1 Introduction

The action for topologically massive gravity consists of the action for cosmological gravity and the Chern-Simons term. Clément, in his search for particle like solutions for this theory, reduced the action [12, 13, 14] to the second order degenerate Lagrangian density

$$L^C = -\frac{m}{2}\zeta\dot{X}^2 - \frac{2m\Lambda}{\zeta} + \frac{\zeta^2}{2\mu m}\mathbf{X} \cdot (\dot{\mathbf{X}} \times \ddot{\mathbf{X}}) \quad (1)$$

depending on positions \mathbf{X} , velocities $\dot{\mathbf{X}}$ and accelerations $\ddot{\mathbf{X}}$. Here, the inner product is defined by the Lorentzian metric, $\zeta = \zeta(t)$ is a function which allows arbitrary reparametrization of the variable t whereas Λ and $1/2m$ are the cosmological and Einstein gravitational constants, respectively.

In a more recent work [61], Sarioğlu and Tekin considered an action consisting of Einstein-Hilbert, Chern-Simons and Pauli-Fierz terms and, obtained the reduced Lagrangian density

$$L^{ST} = \frac{1}{2} \left[a(\dot{X}^2 + \dot{Y}^2) + \frac{2}{\mu} \dot{\mathbf{Y}} \cdot \ddot{\mathbf{X}} - m^2(\mathbf{Y}^2 + \mathbf{X}^2) \right] \quad (2)$$

by suppressing the spatial part of the theory. Here, a, μ, m are parameters and \mathbf{X}, \mathbf{Y} are the position three-vectors. In the context of higher derivative theories, they also considered Pais-Uhlenbeck oscillator as a nonrelativistic limit. This is described by the nondegenerate Lagrangian density

$$L^{PU}[X] = \frac{1}{2} \left[\ddot{X}^2 - (q^2 + p^2) \dot{X}^2 + p^2 \Omega^2 X^2 \right] \quad (3)$$

where X is a real dynamical variable, p and q are positive real parameters [54]. For the theory of topological massive gravity, and Hamiltonian analysis in the ADM framework we refer to the pioneering works of Deser, Jackiw and Templeton [23, 24].

Our interest in the Sarioğlu-Tekin and Clément Lagrangians will be in the framework of the geometry of dynamical systems generated by second order degenerate Lagrangians. Although, there exists extensive studies [7, 46, 48, 49, 50] on the several aspects of the Hamiltonian formulations of the Pais-Uhlenberg Lagrangian (3), the Hamiltonian formulations of the Sarioğlu-Tekin and Clément Lagrangians are absent in the literature. Sarioğlu-Tekin and Clément Lagrangians are degenerate in the sense of Ostrogradsky. For the degenerate or/and constraint systems, the Legendre transformation is not possible in a straight forward way. To achieve this, one may need to employ the Dirac-Bergmann algorithm [6, 25, 26, 69] or, equivalently, its geometric version Gotay-Nester-Hinds algorithm [30, 31, 32, 33]. Here is an incomplete list [5, 15, 34, 35, 59, 41, 51, 52, 55] for the Legendre transformations of singular or/and constraint higher order Lagrangian systems. We, additionally, refer some recent studies; [18] for the Legendre transformation of higher order Lagrangian systems in terms of Tulczyjew's approach, [11] for the stability problem, [42] for the theory on the jet bundles, and [19, 20] for the detail analysis on the second order Lagrangians whose dependence on the accelerations are linearly and/or affine.

At the beginning of 80s, Skinner and Rusk proposed a unification of Lagrangian and Hamiltonian formalisms on the Whitney product of velocity and momentum phase spaces [66, 67, 68]. Adaptation of the Skinner-Rusk unified formalism for the higher order Lagrangian systems is achieved recently by Prieto Martínez and Román-Roy [56]. In the literature, some other versions of the Skinner-Rusk formalism are also available, for example, a field theoretical version is presented in [8, 72], for Lie groups we refer [16], and for an application to the control theory, see [4].

There are two main goals of the present paper. The first one is to obtain the total Hamiltonian functions, the Hamilton's equations, the Dirac-Poisson brackets for the Clément Lagrangian (1) and the Sarioğlu-Tekin Lagrangian (2). The second goal is to present the Skinner-Rusk unified formalisms of these theories.

To achieve these goals, the paper is organized into three main sections. For the sake of completeness, and in order to widen the spectrum of the potential readers, we shall reserve the following section for some necessary theoretical background. Accordingly, we shall start to the next section by recalling the

Ostrogradsky-Legendre transformation. It will be shown that reparametrization invariant second order Lagrangians must be degenerate and must have zero energy. The Dirac-Bergmann constraint algorithm and construction of the Dirac bracket will be summarized. The following section will be ended with a discussion on the Skinner-Rusk unified formalism.

The last two sections, namely 3 and 4, will be devoted for the investigations on the Sarioğlu-Tekin and Clément Lagrangians, respectively. For these sections, the itinerary maps that we shall follow are almost the same. At first, we shall identify the configuration spaces, tangent and cotangent bundles. Then, the associated energy functions will be written. After introducing the primary sets of constraints, the total Hamiltonian function will be written and the Dirac-Bergmann algorithm will be run in order to identify the final constraint submanifold. In each step of the algorithm, we shall revise the total Hamiltonian by adding the secondary constraints. Once the final constraint set is determined, it is immediate to write the Hamilton's equations. This is the first and most common way. An alternative way arriving at the Hamilton's equations is to construct the Dirac bracket. To do this, we shall first classify the constraints, determining the final constraint submanifold, into two classes, namely the first and the second. Then, using this classification, we shall define the Dirac brackets associated with the physical systems. Finally, we shall exhibit the Skinner-Rusk unified formalisms of the Clément and Sarioğlu-Tekin Lagrangians. To do this, the Gotay-Nester-Hinds algorithm will be employed on the associated Whitney bundles.

2 Hamiltonian analysis of the second order Lagrangians

Let M be an m -dimensional configuration manifold M with local coordinates $\mathbf{X} = (X^1, \dots, X^m)$. The velocity phase space of the system is $2m$ -dimensional manifold and it is the tangent bundle TM of M with the induced coordinates $(\mathbf{X}, \dot{\mathbf{X}})$, [1]. The second order tangent bundle T^2M additionally includes the accelerations $\ddot{\mathbf{X}}$ hence it can be equipped with a coordinate system $(\mathbf{X}, \dot{\mathbf{X}}, \ddot{\mathbf{X}})$, [35, 44, 70]. We fix the notation $[\mathbf{X}]$ in order to represent three vectors $(\mathbf{X}, \dot{\mathbf{X}}, \ddot{\mathbf{X}})$. The third order tangent bundle T^3M carries the local coordinates $(\mathbf{X}, \dot{\mathbf{X}}, \ddot{\mathbf{X}}, \dddot{\mathbf{X}})$. If TTM is equipped with the coordinates $(\mathbf{X}, \mathbf{V}, \dot{\mathbf{X}}, \dot{\mathbf{V}})$, then we can arrive at the second iterated bundle T^2M through the identification $\mathbf{V} = \dot{\mathbf{X}}$.

2.1 Jacobi-Ostrogradsky momenta

The history of the theory of Hamiltonian formulations of the higher order Lagrangian systems dated back to more than 150 years ago to the pioneering work of Ostrogradsky [53].

A second order Lagrangian density $L[\mathbf{X}] = L(\mathbf{X}, \dot{\mathbf{X}}, \ddot{\mathbf{X}})$ is a function on the

second order tangent bundle T^2M . The functional differential

$$d(L[\mathbf{X}]dt) = \left(\frac{\partial L}{\partial \mathbf{X}} \cdot d\mathbf{X} + \frac{\partial L}{\partial \dot{\mathbf{X}}} \cdot d\dot{\mathbf{X}} + \frac{\partial L}{\partial \ddot{\mathbf{X}}} \cdot d\ddot{\mathbf{X}} \right) = \mathcal{E}_{\mathbf{X}}(L[\mathbf{X}]) \cdot d\mathbf{X} + \frac{d}{dt} \theta_L[\mathbf{X}] \quad (4)$$

of $L[\mathbf{X}]$ consists of two terms. The first one is the Euler-Lagrange equations given by

$$\mathcal{E}_{\mathbf{X}}(L[\mathbf{X}]) \equiv \frac{\partial L}{\partial \mathbf{X}} - \frac{d}{dt} \frac{\partial L}{\partial \dot{\mathbf{X}}} + \frac{d^2}{dt^2} \frac{\partial L}{\partial \ddot{\mathbf{X}}} = 0, \quad (5)$$

and the second term is a boundary term which is the total derivative of the Lagrangian one-form

$$\theta_L[\mathbf{X}] \equiv \left(\frac{\partial L}{\partial \dot{\mathbf{X}}} - \frac{d}{dt} \frac{\partial L}{\partial \ddot{\mathbf{X}}} \right) \cdot d\mathbf{X} + \frac{\partial L}{\partial \ddot{\mathbf{X}}} \cdot d\dot{\mathbf{X}}. \quad (6)$$

For Lagrangians resulting in the same Euler-Lagrange equations (5), θ_L is not unique. However, its functional exterior derivative

$$\Omega_L[\mathbf{X}] \equiv d\theta_L[\mathbf{X}] \quad (7)$$

is a well-defined presymplectic two-form on T^2M .

On the dual picture, the momentum phase space T^*TM is a canonical symplectic manifold with coordinates $(\mathbf{X}, \dot{\mathbf{X}}, \mathbf{P}^0, \mathbf{P}^1)$ hence it is endowed with the canonical Poisson bracket which results in the fundamental Poisson bracket relations

$$\{X^i, P_j^0\} = \{\dot{X}^i, P_j^1\} = \delta_j^i$$

and, all the others are zero. The form of the Lagrangian one-form θ_L in (6) suggests that we can introduce the momenta

$$\mathbf{P}^0[\mathbf{X}] = \frac{\partial L}{\partial \dot{\mathbf{X}}} - \frac{d}{dt} \frac{\partial L}{\partial \ddot{\mathbf{X}}}, \quad \mathbf{P}^1[\mathbf{X}] = \frac{\partial L}{\partial \ddot{\mathbf{X}}}, \quad (8)$$

for a second order Lagrangian as which called as the Jacobi-Ostrogradsky momenta. In this definition, the Lagrangian one-form θ_L turns out to be

$$\theta_L[\mathbf{X}] \equiv \mathbf{P}^0[\mathbf{X}] \cdot d\mathbf{X} + \mathbf{P}^1[\mathbf{X}] \cdot d\dot{\mathbf{X}}.$$

Note that, $\theta_L[\mathbf{X}]$ is the pull back of the canonical (Liouville) one-form

$$\theta_{T^*TM} = \mathbf{P}^0 \cdot d\mathbf{X} + \mathbf{P}^1 \cdot d\dot{\mathbf{X}}$$

on the cotangent bundle T^*TM by the Legendre map,

$$\mathcal{FL} : T^3M \longrightarrow T^*TM : (\mathbf{X}, \dot{\mathbf{X}}, \ddot{\mathbf{X}}, \ddot{\ddot{\mathbf{X}}}) \longrightarrow (\mathbf{X}, \dot{\mathbf{X}}, \mathbf{P}^0, \mathbf{P}^1). \quad (9)$$

2.2 Reparametrization invariant Lagrangians

In this subsection, we discuss the functional obstructions and energy of the reparametrization invariant second order Lagrangians. The definitions of momenta in Eq.(8) may be inspired by the energy conservation for a second order Lagrangian. A conservation law associated with a second order Lagrangian may be obtained by first solving the Euler-Lagrange equations for the partial derivatives $\partial L/\partial \dot{\mathbf{X}}$ and then using them in the expression for the total derivative dL/dt . The resulting conservation law

$$\frac{d}{dt}E_L[\mathbf{X}] = 0, \quad E_L[\mathbf{X}] \equiv \dot{\mathbf{X}} \cdot \left(\frac{\partial L}{\partial \dot{\mathbf{X}}} - \frac{d}{dt} \frac{\partial L}{\partial \ddot{\mathbf{X}}} \right) + \ddot{\mathbf{X}} \cdot \frac{\partial L}{\partial \ddot{\mathbf{X}}} - L[\mathbf{X}] \quad (10)$$

is a generalization of the well-known definition of the canonical energy for first order Lagrangians. Using the definitions in Eq.(8) we have the expression

$$E_L[\mathbf{X}] = \dot{\mathbf{X}} \cdot \mathbf{P}^0[\mathbf{X}] + \ddot{\mathbf{X}} \cdot \mathbf{P}^1[\mathbf{X}] - L[\mathbf{X}] \quad (11)$$

for the energy function. The same idea works for Lagrangians of any finite order.

Following [58], let us show that the Lagrangians invariant under reparametrization of curves $t \mapsto \mathbf{X}(t)$ are necessarily degenerate and have zero energy. More precisely, we introduce new parametrization $\tau = \tau(t)$, and let $\lambda \equiv dt/d\tau$, $\nu \equiv d\lambda/d\tau = d^2t/d\tau^2$. A second order Lagrangian is reparametrization invariant if

$$\lambda L(\mathbf{X}, \dot{\mathbf{X}}, \ddot{\mathbf{X}}) = L(\mathbf{X}, \lambda \dot{\mathbf{X}}, \lambda^2 \ddot{\mathbf{X}} + \nu \dot{\mathbf{X}}), \quad (12)$$

[21]. Since for $\lambda = 1$ and $\nu = 0$, we have equal derivatives with respect to both parametrization, we differentiate above equation with respect to λ and ν at $(\lambda, \nu) = (1, 0)$ in order to obtain the infinitesimal invariance conditions

$$L = \dot{\mathbf{X}} \cdot \frac{\partial L}{\partial \dot{\mathbf{X}}} + 2\ddot{\mathbf{X}} \cdot \frac{\partial L}{\partial \ddot{\mathbf{X}}}, \quad \dot{\mathbf{X}} \cdot \frac{\partial L}{\partial \ddot{\mathbf{X}}} = 0 \quad (13)$$

also known as Zermelo conditions [71, 73]. After solving $\partial L/\partial \dot{\mathbf{X}}$ and $\partial L/\partial \ddot{\mathbf{X}}$ in terms of momenta from Eq.(8), and substituting these into the first condition in Eq.(13) we arrive that the Lagrangian must be in form

$$L = \dot{\mathbf{X}} \cdot (\mathbf{P}^0 + \dot{\mathbf{P}}^1) + 2\ddot{\mathbf{X}} \cdot \mathbf{P}^1 = \dot{\mathbf{X}} \cdot \mathbf{P}^0 + \ddot{\mathbf{X}} \cdot \mathbf{P}^1,$$

which results with that the energy function E_L given in (11) is zero. Differentiating the second condition in Eq.(13) with respect to $\ddot{\mathbf{X}}$, we obtain a system of equations for $\ddot{\mathbf{X}}$ for which existence of non-zero solutions implies the degeneracy

$$\det \text{Hess}(L) \equiv \det \left[\frac{\partial^2 L}{\partial \ddot{\mathbf{X}}^2} \right] = 0$$

of the second order Lagrangian [40]. We remark here also that, from the second condition in Eq.(13), we arrive a condition $\dot{\mathbf{X}} \cdot \mathbf{P}^1 = 0$ and its differential $\ddot{\mathbf{X}} \cdot \mathbf{P}^1 = -\dot{\mathbf{X}} \cdot \dot{\mathbf{P}}^1$.

2.3 Dirac-Bergmann algorithm

Consider a second order Lagrangian density $L = L[\mathbf{X}]$. As discussed previously, the resulting Euler-Lagrange equations are singular, that is, not all second derivatives are solvable, if the Hessian matrix $\partial^2 L / \partial \ddot{X}^2$ has rank $r < n$. That means there are only $n - r$ independent equations for derivatives higher than second order. The Jacobi-Ostrogradsky momenta \mathbf{P}^0 and \mathbf{P}^1 become functions of $(\mathbf{X}, \dot{\mathbf{X}}, \ddot{\mathbf{X}})$ and $(\mathbf{X}, \dot{\mathbf{X}})$, respectively. More specifically, \mathbf{P}^0 is a linear function of $\ddot{\mathbf{X}}$ and the Euler-Lagrange equations are of third order. From the definition of \mathbf{P}^1 we obtain relations

$$\Phi_\alpha(\mathbf{X}, \dot{\mathbf{X}}, \mathbf{P}^1) \approx 0, \quad \alpha = 1, \dots, n - r$$

among phase space coordinates, called primary constraints [25, 26]. From the equations defining the momenta \mathbf{P}^0 , only some of the second order derivatives are solvable. It is also possible to define constraints from definition of \mathbf{P}^0 if there are equations independent of second derivatives. However, as it is shown in [63] all such possible constraints can be derived from the preservation of primary constraints introduced immediately.

In fact, for any Lagrangian in which the second derivative term is expressed as a triple product (more generally, is involved in completely antisymmetric tensor), only two components of second derivatives can be solved from definition of momenta \mathbf{P}^0 , the last one is satisfied identically. This dimensional degeneracy is the reason behind definition of some constraints by means of dot product.

The equality in the definition of primary constraints is weak in the sense that it is ignored during set up of Dirac formalism, and will actually vanish in any solutions to equations of motion. In other words, Φ_α are not identically zero on the phase space but vanish on primary constraint submanifold they define. The dynamics on this submanifold is not well-defined by the canonical Hamiltonian function

$$H = \mathbf{P}^0 \cdot \dot{\mathbf{X}} + \mathbf{P}^1 \cdot \ddot{\mathbf{X}} - L[\mathbf{X}],$$

it is rather governed by the total Hamiltonian

$$H_T = H + u^\alpha \Phi_\alpha$$

which contains linear combinations of primary constraints with Lagrange multipliers u^α . The requirement that the solutions of Euler-Lagrange equations remain on constraint submanifold is described by the weak equality

$$\dot{\Phi}_\beta = \{\Phi_\beta, H\} + u^\alpha \{\Phi_\beta, \Phi_\alpha\} \approx 0, \quad \beta = 1, \dots, n - r, \quad (14)$$

that is, modulo primary constraints. These consistency conditions may lead to determination of Lagrange multipliers if the left hand sides contain u^α . In this case, one solves for u^α through the set of linear equations

$$\{\Phi_\beta, \Phi_\alpha\} u^\alpha = -\{\Phi_\beta, H\}$$

for which the solution set, namely, number of multipliers that can be solved is characterized by the rank of the skew-symmetric matrix $\{\Phi_\beta, \Phi_\alpha\}$ of Poisson

brackets. Obviously, if the number of primary constraints is odd u^α s cannot be solved completely and one aspects more constraints to determine H_T in terms of phase space variables. This secondary constraints follow if left hand sides does not contain u^α or, $n-r$ is odd. Repeating this process, one enlarges the primary constraint set with the new (secondary, tertiary, etc.) constraints, redefines H_T by introducing new Lagrange multipliers for new constraints and, repeats the consistency computations.

Iterated applications of consistency computations lead to a complete set of constraints $\Phi_\alpha : \alpha = 1, \dots, k$. Let

$$\mathcal{M}_{\alpha\beta} = \{\Phi_\alpha, \Phi_\beta\}$$

be the matrix of Poisson brackets of constraints modulo all constraints. If $\text{rank}(\mathcal{M}_{\alpha\beta}) = r$, then $\ker(\mathcal{M}_{\alpha\beta})$ is $(k-r)$ -dimensional. A basis for the kernel can be constructed from linear combinations ψ_α of Φ_α satisfying

$$\{\psi_\alpha, \psi_\beta\} \approx 0, \quad \alpha, \beta = 1, \dots, k-r$$

and are called first class constraints. Note that the number of Lagrange multipliers which can be solved is also determined by the matrix of all constraints. Let $\chi_\alpha : \alpha = 1, \dots, r$ be the second class constraints whose Poisson brackets does not vanish (modulo constraints). Define the $r \times r$ -matrix

$$C_{\alpha\beta} = \{\chi_\alpha, \chi_\beta\}, \quad \alpha, \beta = 1, \dots, r$$

which is invertible by construction. Define the Dirac bracket

$$\{f, g\}_{DB} = \{f, g\} - \{f, \chi_\alpha\}(C^{-1})^{\alpha\beta}\{\chi_\beta, g\} \quad (15)$$

[69]. Note that, since $\{f, \chi_\alpha\}_{DB} = 0$ for arbitrary function f , second class constraints can be set to zero either before or after evaluation of Dirac bracket. The initial $2n$ -dimensional Hamiltonian system with $k-r$ -first class and r -second class constraints becomes $2n - 2(k-r) - r = 2n - 2k + r$ -dimensional reduced Hamiltonian system for the Dirac bracket and with the total Hamiltonian function of Dirac. The final bracket eliminates the second class constraints from the set of all constraints leaving a complete set of first class constraints. First class constraints form a closed local symmetry algebra for the system. Computing

$$\{\psi_\alpha, H\} = c_\alpha^\beta \psi_\beta, \quad \{\psi_\alpha, \psi_\beta\} = c_{\alpha\beta}^\gamma \psi_\gamma$$

one finds the structure constants of this algebra [25], [26].

2.4 Skinner-Rusk unified formalism

The Skinner-Rusk unified formalism is to define a proper submanifold of the presymplectic Pontryagin bundle, Whitney product of velocity and momentum phase spaces, which enables one to study the Hamiltonian and Lagrangian formalism altogether [66, 67, 68]. By following [56], let us summarize the Skinner-Rusk unified formalism in the case of second order Lagrangians [56]. We refer [57] for the non-autonomous cases.

Consider the second order Pontryagin bundle

$$P^3Q = T^3Q \times_{TQ} T^*TQ \quad (16)$$

which is the Whitney product of the third order tangent bundle T^3Q and the iterated cotangent bundle T^*TQ over the base manifold TQ . The induced coordinates on P^3Q is given by six-tuples

$$(\mathbf{X}, \dot{\mathbf{X}}, \ddot{\mathbf{X}}, \ddot{\mathbf{X}}, \mathbf{P}^0, \mathbf{P}^1) \in P^3Q \quad (17)$$

obtained those defined on T^3Q and T^*TQ . There are projections pr_1 and pr_2 from P^3Q to T^3Q and T^*TQ , respectively. Skinner-Rusk formalism on the second order bundle is to search a possible solution of the presymplectic Hamilton's equation

$$i_{X_{P^3Q}} \Omega_{P^3Q} = -dE, \quad (18)$$

where the Hamiltonian function E is assumed to be the energy function (11) in form

$$E(\mathbf{X}, \dot{\mathbf{X}}, \ddot{\mathbf{X}}, \ddot{\mathbf{X}}, \mathbf{P}^0, \mathbf{P}^1) = \mathbf{P}^0 \cdot \dot{\mathbf{X}} + \mathbf{P}^1 \cdot \ddot{\mathbf{X}} - \mathbf{L}(\mathbf{X}, \dot{\mathbf{X}}, \ddot{\mathbf{X}}). \quad (19)$$

Here, Ω_{P^3Q} is the presymplectic two-form on P^3Q and obtained by pull-back of the canonical symplectic two-form Ω_{T^*TQ} on T^*TQ by the projection pr_2 .

In the local chart (17), Ω_{P^3Q} is computed to be

$$\Omega_{P^3Q} = (pr_2)^* \Omega_{T^*TQ} = d\mathbf{P}^0 \wedge d\mathbf{X} + d\mathbf{P}^1 \wedge d\dot{\mathbf{X}}, \quad (20)$$

whereas the Hamiltonian vector field looks like

$$X_{P^3M} = \dot{\mathbf{X}} \cdot \nabla_{\mathbf{X}} + \ddot{\mathbf{X}} \cdot \nabla_{\dot{\mathbf{X}}} + \ddot{\mathbf{X}} \cdot \nabla_{\ddot{\mathbf{X}}} + \mathbf{B} \cdot \nabla_{\ddot{\mathbf{X}}} + \nabla_{\mathbf{X}} L \cdot \nabla_{\mathbf{P}^0} + (\nabla_{\dot{\mathbf{X}}} L - \mathbf{P}^0) \cdot \nabla_{\mathbf{P}^1} \quad (21)$$

with compatibility conditions $\mathbf{P}^1 = \nabla_{\ddot{\mathbf{X}}} L$ to be sure that X_{P^3M} is tangent to W_0 (c.f. the second Jacobi Ostrogradsky momenta (8)). This assumption is necessary to guarantee that the projection

$$X_{T^3M} = (pr_1)_* X_{P^3M}$$

is a Euler-Lagrange vector field, that is the Hamilton's equations

$$i_{X_{T^3M}} \Omega_{T^3M} = -dE_L$$

give Euler-Lagrange equations on the submanifold $pr_1(W_f)$ of T^3M .

Finding a vector field X_{P^3M} satisfying Hamilton's equations (18) is possible on a submanifold W_f , so called final constraint submanifold, of P^3M . We start with determining the primary constraint submanifold W_0 by defining the primary constraints

$$\Psi = \mathbf{P}^0 - \nabla_{\dot{\mathbf{X}}} L + \frac{d}{dt} \nabla_{\ddot{\mathbf{X}}} L, \quad \Phi = \mathbf{P}^1 - \nabla_{\ddot{\mathbf{X}}} L. \quad (22)$$

If the tangency conditions

$$X_{P^3M}(\Psi) = \mathbf{0} \quad \text{and} \quad X_{P^3M}(\Phi) = \mathbf{0} \quad (23)$$

hold, then the final constraint submanifold W_f equals to the primary constraint submanifold W_0 . This occurs if the Lagrangian is regular, that is the tangent map of the Jacobi-Ostrogradsky momenta (8) is surjective submersion at every point in its domain. If the Lagrangian is degenerate, then the tangency conditions (23) lead to two new sets of constraints

$$\Psi_1 = X_{P^3M}\Psi, \quad \Phi_2 = X_{P^3M}\Phi.$$

and, correspondingly, a constraint submanifold W_1 of W_0 by additionally requiring $\Psi_1 = \Phi_1 = \mathbf{0}$. If we ask the tangency conditions

$$X_{P^3M}(\Psi_1) = \mathbf{0} \quad \text{and} \quad X_{P^3M}(\Phi_1) = \mathbf{0}$$

for the constraints then there are possible scenarios. The first one is to observe that $X_{P^3M}\Psi_1$ and $X_{P^3M}\Phi_1$ are identically zero. This gives that W_1 is the final submanifold W_f and we are done. The second one is to arrive at two new set of constraints

$$\Psi_2 = X_{P^3M}(\Psi_1), \quad \Phi_2 = X_{P^3M}\Phi_1,$$

called the first-generation secondary constraints. Using these constraints, define a submanifold W_2 of W_1 by additionally requiring $\Psi_2 = \Phi_2 = 0$. Repeating this algorithm, we may obtain k -generation secondary constraints which defines a submanifold W_k . If the nested sequence

$$W_k \subset W_{k-1} \subset \dots \subset W_0$$

has an end, that is $W_{k+1} = W_k$, then W_k is the final constraint submanifold and on this submanifold and the vector field X_{P^3M} has a well-defined expression satisfying Eq.(18).

3 Hamiltonian analysis of Sarioğlu-Tekin Lagrangian

3.1 Sarioğlu-Tekin Lagrangian

We start with a 6-dimensinal manifold Q with local coordinates (\mathbf{X}, \mathbf{Y}) consisting of two 3-dimensional vectors. The higher order tangent bundles are equipped with the following induced sets of coordinates

$$\begin{aligned} (\mathbf{X}, \mathbf{Y}, \dot{\mathbf{X}}, \dot{\mathbf{Y}}) &\in TQ \\ (\mathbf{X}, \mathbf{Y}, \dot{\mathbf{X}}, \dot{\mathbf{Y}}, \ddot{\mathbf{X}}, \ddot{\mathbf{Y}}) &\in T^2Q \\ (\mathbf{X}, \mathbf{Y}, \dot{\mathbf{X}}, \dot{\mathbf{Y}}, \ddot{\mathbf{X}}, \ddot{\mathbf{Y}}, \dddot{\mathbf{X}}, \dddot{\mathbf{Y}}) &\in T^3Q. \end{aligned}$$

In [61], Sarioğlu and Tekin proposed a degenerate second order Lagrangian on T^2Q given by

$$L^{ST}[\mathbf{X}, \mathbf{Y}] = \frac{1}{2} \left[a(\dot{X}^2 + \dot{Y}^2) + \frac{2}{\mu} \dot{\mathbf{Y}} \cdot \ddot{\mathbf{X}} - m^2(Y^2 + X^2) \right]. \quad (24)$$

In this case, the second order Euler-Lagrange equations (5) take the particular form

$$m^2\mathbf{X} + a\ddot{\mathbf{X}} = \frac{1}{\mu}\mathbf{Y}^{(3)}, \quad m^2\mathbf{Y} + a\ddot{\mathbf{Y}} = -\frac{1}{\mu}\mathbf{X}^{(3)}. \quad (25)$$

The Lagrangian one-form (6) takes the particular form

$$\theta_L = (a\dot{\mathbf{X}} - \frac{1}{\mu}\ddot{\mathbf{Y}}) \cdot d\mathbf{X} + (a\dot{\mathbf{Y}} + \frac{1}{\mu}\ddot{\mathbf{X}}) \cdot d\mathbf{Y} + \frac{1}{\mu}\dot{\mathbf{Y}} \cdot d\dot{\mathbf{X}},$$

on the second order tangent bundle T^2Q whereas the exterior derivative of θ_L becomes

$$\Omega_L = a(d\dot{\mathbf{X}} \wedge d\mathbf{X} + d\dot{\mathbf{Y}} \wedge d\mathbf{Y}) + \frac{1}{\mu}d\dot{\mathbf{Y}} \wedge d\dot{\mathbf{X}} + \frac{1}{\mu}(d\ddot{\mathbf{X}} \wedge d\mathbf{Y} - d\ddot{\mathbf{Y}} \wedge d\mathbf{X}).$$

Here, we use the abbreviation $\dot{\wedge}$ defined as

$$d\mathbf{X} \dot{\wedge} d\mathbf{Y} = dX^1 \wedge dY^1 + dX^2 \wedge dY^2 + dX^3 \wedge dY^3. \quad (26)$$

In this notation, $d\mathbf{X} \dot{\wedge} d\mathbf{X} = 0$ identically.

The Sarioğlu-Tekin Lagrangian (24) has $SO(3)$ invariance resulting in the momenta

$$\frac{1}{2}J[\mathbf{X}, \mathbf{Y}] = a\mathbf{Y} \times \dot{\mathbf{Y}} + \frac{1}{\mu}\mathbf{Y} \times \ddot{\mathbf{X}} + a\mathbf{X} \times \dot{\mathbf{X}} - \frac{1}{\mu}\mathbf{X} \times \ddot{\mathbf{Y}} + \frac{1}{\mu}\dot{\mathbf{X}} \times \dot{\mathbf{Y}}$$

and the time-translation invariance gives the energy

$$E^{ST}[\mathbf{X}, \mathbf{Y}] = a(\dot{X}^2 + \dot{Y}^2) + \frac{2}{\mu}(\dot{\mathbf{Y}} \cdot \ddot{\mathbf{X}} - \dot{\mathbf{X}} \cdot \ddot{\mathbf{Y}}) + m^2(Y^2 + X^2)$$

both of which may be shown to satisfy the conservation laws $\dot{J} = \dot{E}^{ST} = 0$ via Euler-Lagrange equations.

3.2 Dirac-Bergmann Algorithm

The iterated cotangent bundle T^*TQ is 24-dimensional and equipped with a local chart

$$(\mathbf{X}, \mathbf{Y}, \dot{\mathbf{X}}, \dot{\mathbf{Y}}, \mathbf{P}_X^0, \mathbf{P}_Y^0, \mathbf{P}_X^1, \mathbf{P}_Y^1) \in T^*TQ.$$

The Jacobi-Ostrogradsky momenta (8) are defined by

$$\mathbf{P}_X^0 = a\dot{\mathbf{X}} - \frac{1}{\mu}\ddot{\mathbf{Y}}, \quad \mathbf{P}_X^1 = \frac{1}{\mu}\dot{\mathbf{Y}}, \quad \mathbf{P}_Y^0 = a\dot{\mathbf{Y}} + \frac{1}{\mu}\ddot{\mathbf{X}}, \quad \mathbf{P}_Y^1 = \mathbf{0}, \quad (27)$$

whereas the canonical Hamiltonian functions is

$$H^{ST} = \mathbf{P}_X^0 \cdot \dot{\mathbf{X}} + \mathbf{P}_Y^0 \cdot \dot{\mathbf{Y}} - \frac{a}{2}(\dot{X}^2 + \dot{Y}^2) + \frac{m^2}{2}(X^2 + Y^2). \quad (28)$$

The momenta (27) give the following two second derivatives

$$\ddot{\mathbf{Y}} = \mu(a\dot{\mathbf{X}} - \mathbf{P}_X^0), \quad \ddot{\mathbf{X}} = \mu(\mathbf{P}_Y^0 - a\dot{\mathbf{Y}}), \quad (29)$$

along with and the set of primary constraints

$$\Phi = \mathbf{P}_X^1 - \frac{1}{\mu}\dot{\mathbf{Y}} \approx \mathbf{0}, \quad \Psi = \mathbf{P}_Y^1 \approx \mathbf{0}. \quad (30)$$

We define the total Hamiltonian

$$H_T^{ST} = H^{ST} + \mathbf{U} \cdot \Phi + \mathbf{V} \cdot \Psi$$

as the sum of the canonical Hamiltonian H^{ST} in (28) and the primary constraints Φ and Ψ in (30) multiplied by the Lagrange multipliers \mathbf{U} and \mathbf{V} , respectively. The consistency check results with determination of the Lagrange multipliers

$$\mathbf{V} = \mu(a\dot{\mathbf{X}} - \mathbf{P}_X^0), \quad \mathbf{U} = \mu(\mathbf{P}_Y^0 - a\dot{\mathbf{Y}}) \quad (31)$$

without causing a new constraint. This means that we have arrived the final submanifold.

After the substitution of the Lagrange multipliers \mathbf{V} and \mathbf{U} in (31) into the total Hamiltonian H_T^{ST} , the total turns out to be

$$\begin{aligned} H_T^{ST} = & \mu(\mathbf{P}_Y^0 \cdot \mathbf{P}_X^1 - \mathbf{P}_X^0 \cdot \mathbf{P}_Y^1) + a\mu(\dot{\mathbf{X}} \cdot \mathbf{P}_Y^1 - \dot{\mathbf{Y}} \cdot \mathbf{P}_X^1) + \mathbf{P}_X^0 \cdot \dot{\mathbf{X}} \\ & - \frac{a}{2}(\dot{X}^2 - \dot{Y}^2) + \frac{m^2}{2}(X^2 + Y^2). \end{aligned} \quad (32)$$

Note that, to arrive at the total Hamiltonian function (32), we may follow a more direct way by solving $\ddot{\mathbf{X}}$ and $\ddot{\mathbf{Y}}$ from the equations (27) and substituting $\ddot{\mathbf{X}}$ and $\ddot{\mathbf{Y}}$ in Eqs.(27) into the canonical Hamiltonian function without referring any constraint analysis. For the base variables $(\mathbf{X}, \mathbf{Y}, \dot{\mathbf{X}}, \dot{\mathbf{Y}})$, the equations of motion governed by the total Hamiltonian H_T^{ST} are

$$\dot{\mathbf{X}} = \dot{\mathbf{X}} - \mu\mathbf{P}_Y^1, \quad \ddot{\mathbf{X}} = \mu\mathbf{P}_Y^0 - a\mu\dot{\mathbf{Y}}, \quad \dot{\mathbf{Y}} = \mu\mathbf{P}_X^1, \quad \ddot{\mathbf{Y}} = -\mu\mathbf{P}_X^0 + a\mu\dot{\mathbf{X}} \quad (33)$$

which are satisfied identically on the constraint submanifold. For momenta $(\mathbf{P}_X^0, \mathbf{P}_Y^0, \mathbf{P}_X^1, \mathbf{P}_Y^1)$, the equations of motion are

$$\dot{\mathbf{P}}_X^0 = -m^2\mathbf{X}, \quad \dot{\mathbf{P}}_X^1 = a\dot{\mathbf{X}} - a\mu\mathbf{P}_Y^1 - \mathbf{P}_X^0, \quad \dot{\mathbf{P}}_Y^0 = -m^2\mathbf{Y}, \quad \dot{\mathbf{P}}_Y^1 = -a\dot{\mathbf{Y}} + a\mu\mathbf{P}_X^1, \quad (34)$$

where the second and the fourth ones are identically satisfied. The first and third equations give Euler-Lagrange equations (25) only after the substitution of the second order equations (29).

3.3 Dirac-Poisson Bracket

The constraints Φ and Ψ are all second class, hence the Poisson brackets of them define the nondegenerate 6×6 constraint matrix

$$C = \begin{pmatrix} \{\Phi_i, \Phi_j\} & \{\Phi_i, \Psi_j\} \\ \{\Psi_j, \Phi_i\} & \{\Psi_i, \Psi_j\} \end{pmatrix} = \frac{1}{\mu} \begin{pmatrix} \mathbf{0} & -\mathbb{I} \\ \mathbb{I} & \mathbf{0} \end{pmatrix}.$$

Hence, the Dirac bracket in Eq.(15) takes the particular form

$$\{F, G\}_{DB} = \{F, G\} - \{F, \Phi_i\} \mu \delta^{ij} \{\Psi_j, G\} + \{F, \Psi_i\} \mu \delta^{ij} \{\Phi_j, G\} \quad (35)$$

and results in the reduced Poisson structure

$$\{X^i, (P_X^0)_j\}_{DB} = \{\dot{X}^i, (P_X^1)_j\}_{DB} = \{Y^i, (P_Y^0)_j\}_{DB} = \delta_j^i, \quad \{\dot{X}^i, \dot{Y}^j\}_{DB} = \mu \delta^{ij}. \quad (36)$$

It is straight-forward to check that, using the Dirac bracket (35), the equations of motion generated by the canonical Hamiltonian H^{ST} in Eq.(28) is exactly equal to the dynamics generated by the H_T^{ST} .

3.4 Skinner-Rusk unified formalism

On the Pontryagin bundle $P^3Q = T^3Q \times_{TQ} T^*TQ$, the presymplectic two-form Ω_{P^3Q} defined in Eq.(20), and the canonical Hamiltonian function defined in (19) turn out to be

$$\Omega_{P^3Q} = pr_2^* \Omega_{T^*TQ} = d\mathbf{P}_X^0 \wedge d\mathbf{X} + d\mathbf{P}_Y^0 \wedge d\mathbf{Y} + d\mathbf{P}_X^1 \wedge d\dot{\mathbf{X}} + d\mathbf{P}_Y^1 \wedge d\dot{\mathbf{Y}} \quad (37)$$

whereas the canonical Hamiltonian function

$$H_{P^3Q} = \mathbf{P}_X^0 \cdot \dot{\mathbf{X}} + \mathbf{P}_Y^0 \cdot \dot{\mathbf{Y}} + \mathbf{P}_X^1 \cdot \ddot{\mathbf{X}} + \mathbf{P}_Y^1 \cdot \ddot{\mathbf{Y}} - L^{ST}[\mathbf{X}, \mathbf{Y}].$$

To determine a unique vector field X_{P^3Q} satisfying the Hamilton's equations (18), we start with

$$\begin{aligned} X_{P^3Q} &= \dot{\mathbf{X}} \cdot \nabla_{\mathbf{X}} + \ddot{\mathbf{X}} \cdot \nabla_{\dot{\mathbf{X}}} + \ddot{\mathbf{X}} \cdot \nabla_{\ddot{\mathbf{X}}} + \mathbf{K}_X \cdot \nabla_{\ddot{\mathbf{X}}} + \dot{\mathbf{Y}} \cdot \nabla_{\mathbf{Y}} + \ddot{\mathbf{Y}} \cdot \nabla_{\dot{\mathbf{Y}}} + \ddot{\mathbf{Y}} \cdot \nabla_{\ddot{\mathbf{Y}}} \\ &\quad + \mathbf{K}_Y \cdot \nabla_{\ddot{\mathbf{Y}}} - m^2 \mathbf{X} \cdot \nabla_{\mathbf{P}_X^0} - m^2 \mathbf{Y} \cdot \nabla_{\mathbf{P}_Y^0} + \left(a \dot{\mathbf{X}} - \mathbf{P}_X^0\right) \cdot \nabla_{\mathbf{P}_X^1} \\ &\quad + \left(a \dot{\mathbf{Y}} - \mathbf{P}_Y^0 + \frac{1}{\mu} \ddot{\mathbf{X}}\right) \cdot \nabla_{\mathbf{P}_Y^1}, \end{aligned} \quad (38)$$

where we have two sets of unknown coefficient functions \mathbf{K}_X and \mathbf{K}_Y . The graph of the Legendre transformation \mathcal{FL}^{ST} is described by the following primary constraints

$$\begin{aligned} \bar{\Phi} &= \mathbf{P}_X^0 - a \dot{\mathbf{X}} + \frac{1}{\mu} \ddot{\mathbf{Y}}, \quad \bar{\Psi} = \mathbf{P}_Y^0 - a \dot{\mathbf{Y}} + \frac{1}{\mu} \ddot{\mathbf{X}} \\ \Phi &= \mathbf{P}_X^1 - \frac{1}{\mu} \dot{\mathbf{Y}}, \quad \Psi = \mathbf{P}_Y^1. \end{aligned}$$

The first step is to check the tangency conditions:

$$\begin{aligned}
X_{P^3N} \bar{\Phi} &= \bar{\Phi}_1 = \frac{1}{\mu} \ddot{\mathbf{Y}} - m^2 \mathbf{X} - a \ddot{\mathbf{X}}, \\
X_{P^3N} \bar{\Psi} &= \bar{\Psi}_1 = \frac{1}{\mu} \ddot{\mathbf{X}} + m^2 \mathbf{Y} + a \ddot{\mathbf{Y}}, \\
X_{P^3N} \Phi &= -\bar{\Phi}, \\
X_{P^3N} \Psi &= -\bar{\Psi}.
\end{aligned} \tag{39}$$

The last two equations are weakly zero, so that we only take the first two $\bar{\Phi}_1$ and $\bar{\Psi}_1$ as new constraints. The first constraint submanifold W_1 is defined by the set of functions $\bar{\Phi}, \bar{\Psi}, \Phi, \Psi, \bar{\Phi}_1, \bar{\Psi}_1$. For the secondary constraints, we have that

$$\begin{aligned}
X_{P^3N} \bar{\Phi}_1 &= \frac{1}{\mu} \mathbf{K}_Y - m^2 \dot{\mathbf{X}} - a \ddot{\mathbf{X}} \\
X_{P^3N} \bar{\Psi}_1 &= -\frac{1}{\mu} \mathbf{K}_X - m^2 \dot{\mathbf{Y}} - a \ddot{\mathbf{Y}}
\end{aligned}$$

from which, by requiring that they weakly equal to zero, we obtain unknown functions

$$\begin{aligned}
\mathbf{K}_Y &= \mu m^2 \dot{\mathbf{X}} + \mu a \ddot{\mathbf{X}} \\
\mathbf{K}_X &= -\mu m^2 \dot{\mathbf{Y}} - \mu a \ddot{\mathbf{Y}}.
\end{aligned}$$

We have no secondary constraints and that $W_1 = W_f$ is the final constraint submanifold.

We obtain the Euler-Lagrange vector field X_{T^3Q} on T^3Q by projecting X_{P^3Q} in (38) via pr_1 , that is

$$\begin{aligned}
X_{T^3N} &= \dot{\mathbf{X}} \cdot \nabla_{\mathbf{X}} + \ddot{\mathbf{X}} \cdot \nabla_{\dot{\mathbf{X}}} + \ddot{\mathbf{X}} \cdot \nabla_{\ddot{\mathbf{X}}} - \left(\mu m^2 \dot{\mathbf{Y}} + \mu a \ddot{\mathbf{Y}} \right) \cdot \nabla_{\ddot{\mathbf{X}}} + \\
&\quad + \dot{\mathbf{Y}} \cdot \nabla_{\mathbf{Y}} + \ddot{\mathbf{Y}} \cdot \nabla_{\dot{\mathbf{Y}}} + \ddot{\mathbf{Y}} \cdot \nabla_{\ddot{\mathbf{Y}}} + \left(\mu m^2 \dot{\mathbf{X}} + \mu a \ddot{\mathbf{X}} \right) \cdot \nabla_{\ddot{\mathbf{Y}}}.
\end{aligned} \tag{40}$$

The energy $E_{T^3Q} = (pr_1)^* E_{P^3Q}$ is given by

$$E_{T^3Q} = \left(a \dot{\mathbf{X}} - \frac{1}{\mu} \ddot{\mathbf{Y}} \right) \cdot \dot{\mathbf{X}} + \left(a \dot{\mathbf{Y}} + \frac{2}{\mu} \ddot{\mathbf{X}} \right) \cdot \dot{\mathbf{Y}} - L^{ST}, \tag{41}$$

and it satisfies the Hamilton's equations

$$i_{X_{T^3Q}} \Omega_{T^3Q} = -dE_{T^3Q}$$

on $S_f = pr^1(W_f)$ if the Euler-Lagrange equations (25) hold. Here, Ω_{T^3Q} is the two-form obtained by the push forward of Ω_{P^3Q} in Eq.(37) to T^3Q .

4 Hamiltonian analysis of Clément Lagrangian

4.1 Clément Lagrangian

Let us record here Clément's degenerate second order Lagrangian density

$$L^C[\mathbf{X}] = -\frac{m}{2}\zeta\dot{X}^2 - \frac{2m\Lambda}{\zeta} + \frac{\zeta^2}{2\mu m}\mathbf{X} \cdot (\dot{\mathbf{X}} \times \ddot{\mathbf{X}}) \quad (42)$$

on the second order tangent bundle T^2M with local coordinates $[\mathbf{X}] = (\mathbf{X}, \dot{\mathbf{X}}, \ddot{\mathbf{X}})$. Here, the inner product $X^2 = T^2 - X^2 - Y^2$ is defined by the Lorentzian metric and the triple product is $\mathbf{X} \cdot (\dot{\mathbf{X}} \times \ddot{\mathbf{X}}) = \epsilon_{ijk} X^i \dot{X}^j \ddot{X}^k$ where ϵ_{ijk} is the completely antisymmetric tensor of rank three. Dot denotes the derivative with respect to the variable t and $\zeta = \zeta(t)$ is a function which allows arbitrary reparametrization of the variable t . Λ and $1/2m$ are cosmological and Einstein gravitational constants, respectively.

The variation of Clément Lagrangian (42) with respect to ζ gives the energy constraint

$$E^C[\mathbf{X}] = -\frac{m}{2}\dot{X}^2 + 2\frac{m\Lambda}{\zeta^2} + \frac{\zeta}{m\mu}\mathbf{X} \cdot (\dot{\mathbf{X}} \times \ddot{\mathbf{X}}) = 0. \quad (43)$$

whereas the variation of the Lagrangian (42) with respect to \mathbf{X} results with the third order Euler-Lagrange equations

$$2m^2\mu\ddot{\mathbf{X}} + 3\dot{\mathbf{X}} \times \ddot{\mathbf{X}} + 2\mathbf{X} \times \mathbf{X}^{(3)} = \mathbf{0}. \quad (44)$$

In the Euler-Lagrange equations (44), we set the reparametrization function ζ equal to one. The Clément Lagrangian (42) is invariant under translations in t and pseudo-rotations in space. Time translation symmetry gives the conservation of energy in Eq.(43).

Following the definitions (6) and (7), we compute the Lagrange one-form $\theta_L[\mathbf{X}]$ and the pre-symplectic two-form $\Omega_L[\mathbf{X}]$ on T^2M as follows

$$\theta_L[\mathbf{X}] = -(m\zeta\dot{\mathbf{X}} + \frac{\zeta^2}{\mu m}\mathbf{X} \times \ddot{\mathbf{X}}) \cdot d\mathbf{X} + \frac{\zeta^2}{2\mu m}(\mathbf{X} \times \dot{\mathbf{X}}) \cdot d\dot{\mathbf{X}} \quad (45)$$

$$\begin{aligned} \Omega_L[\mathbf{X}] = & m\zeta d\dot{\mathbf{X}} \wedge d\mathbf{X} - \frac{\zeta^2}{2\mu m}\dot{\mathbf{X}} \cdot d\dot{\mathbf{X}} \wedge d\mathbf{X} - \frac{\zeta^2}{2\mu m}\mathbf{X} \cdot d\dot{\mathbf{X}} \wedge d\dot{\mathbf{X}} \\ & - \frac{\zeta^2}{\mu m}[\mathbf{X} \cdot d\mathbf{X} \wedge d\ddot{\mathbf{X}} + \ddot{\mathbf{X}} \cdot d\mathbf{X} \wedge d\mathbf{X}], \end{aligned} \quad (46)$$

where \wedge is as defined in (26). If $\mathbf{X} = (X^1, X^2, X^3)$ and $\mathbf{Y} = (Y^1, Y^2, Y^3)$ are two vectors, we define two exterior products

$$\begin{aligned} d\mathbf{X} \wedge d\mathbf{Y} &= (dX^2 \wedge dY^3 - dY^2 \wedge dX^3, dX^3 \wedge dY^1 - dY^3 \wedge dX^1, \\ &\quad dX^1 \wedge dY^2 - dY^1 \wedge dX^2) \\ d\mathbf{X} \wedge d\mathbf{X} &= 2(dX^2 \wedge dX^3, dX^3 \wedge dX^1, dX^1 \wedge dX^2) \neq 0. \end{aligned}$$

The matrix representation of the presymplectic two form Ω_L is given by

$$\Omega_L[\mathbf{X}] = \begin{bmatrix} \frac{1}{\mu}\hat{\mathbf{X}} & -\frac{1}{2}\mathbb{I} - \frac{1}{4\mu}\hat{\mathbf{X}} & \frac{1}{2\mu}\hat{\mathbf{X}} & \mathbf{0} \\ \frac{1}{2}\mathbb{I} + \frac{1}{4\mu}\hat{\mathbf{X}} & \frac{1}{2\mu}\hat{\mathbf{X}} & \mathbf{0} & \mathbf{0} \\ -\frac{1}{2\mu}\hat{\mathbf{X}} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

Here, \mathbb{I} denotes 3×3 identity matrix and we employed the hat map $\hat{\mathbf{X}}$ notation

$$\hat{\cdot}: \mathbb{R}^3 \rightarrow \mathfrak{so}(3) : \mathbf{X} = (X, Y, Z) \rightarrow \hat{\mathbf{X}} = \begin{bmatrix} 0 & -Z & Y \\ Z & 0 & -X \\ -Y & X & 0 \end{bmatrix} \quad (47)$$

which is an isomorphism from \mathbb{R}^3 to the space of skew-symmetric matrices. Note that, the hat map isomorphism can also be seen by the identity $\hat{\mathbf{X}}\mathbf{Y} = \mathbf{X} \times \mathbf{Y}$.

The Clément Lagrangian (42) is invariant under translations in t and pseudo-rotations in space. Time translation symmetry gives the conservation of energy in Eq.(43). Rotational invariance implies the conservation of angular momentum

$$\mathbf{J}[\mathbf{X}] = m\mathbf{X} \times \dot{\mathbf{X}} + \frac{m}{2\mu}[2\mathbf{X} \times (\mathbf{X} \times \ddot{\mathbf{X}}) - \dot{\mathbf{X}} \times (\mathbf{X} \times \dot{\mathbf{X}})]. \quad (48)$$

4.2 Dirac-Bergmann Algorithm

We recall the Darboux' coordinates $(\mathbf{X}, \dot{\mathbf{X}}, \mathbf{P}^0, \mathbf{P}^1)$ and compute the Jacobi-Ostrogradsky momenta

$$\mathbf{P}^0[\mathbf{X}] = -m\zeta\dot{\mathbf{X}} + \frac{\zeta^2}{\mu m}\ddot{\mathbf{X}} \times \mathbf{X}, \quad \mathbf{P}^1[\mathbf{X}] = \frac{\zeta^2}{2\mu m}\mathbf{X} \times \dot{\mathbf{X}}, \quad (49)$$

on momentum phase space T^*TM with the Darboux' coordinates $(\mathbf{X}, \dot{\mathbf{X}}, \mathbf{P}^0, \mathbf{P}^1)$. The Jacobi-Ostrogradsky momenta $(\mathbf{P}^0, \mathbf{P}^1)$ presented in (49) define the Legendre map $\mathcal{FL}: T^3M \rightarrow T^*TM$ whose tangent

$$D\mathcal{FL} = \begin{bmatrix} \mathbb{I} & 0 & 0 & 0 \\ 0 & \mathbb{I} & 0 & 0 \\ \frac{1}{\mu}\hat{\mathbf{X}} & -\mathbb{I} & -\frac{1}{\mu}\hat{\mathbf{X}} & 0 \\ -\frac{1}{2\mu}\hat{\mathbf{X}} & \frac{1}{2\mu}\hat{\mathbf{X}} & 0 & 0 \end{bmatrix}$$

is a 12×12 matrix whose rank is 9. Note that the momenta \mathbf{J} in (48) can be regarded as the pull-back of the angular momentum

$$\mathbf{J} = \mathbf{X} \times \mathbf{P}^0[\mathbf{X}] + \dot{\mathbf{X}} \times \mathbf{P}^1[\mathbf{X}].$$

The canonical Hamiltonian function for the Clément Lagrangian (42) is

$$H^C = \mathbf{P}^0 \cdot \dot{\mathbf{X}} + \mathbf{P}^1 \cdot \ddot{\mathbf{X}} - L^C = \frac{m}{2}\zeta\dot{\mathbf{X}}^2 + \frac{2m\Lambda}{\zeta} + \dot{\mathbf{X}} \cdot \mathbf{P}^0. \quad (50)$$

The pull-back of the canonical Hamiltonian H^C to T^2M by Jacobi-Ostrogradsky momenta corresponds to the energy function $E^C[\mathbf{X}]$ presented in (43). We shall apply Dirac constraint analysis to obtain the Hamiltonian formulation of the dynamics. The Ostrogradsky momenta \mathbf{P}^1 lead to the set Φ of primary constraints

$$\Phi = \mathbf{P}^1 - \frac{\zeta^2}{2\mu m} \mathbf{X} \times \dot{\mathbf{X}}. \quad (51)$$

The consistency conditions

$$\dot{\Phi} = \{\Phi, H^C + \mathbf{U} \cdot \Phi\} = -m\zeta \dot{\mathbf{X}} - \mathbf{P}^0 + \frac{\zeta^2}{\mu m} \mathbf{U} \times \mathbf{X} \approx 0, \quad (52)$$

of the primary constraints for Φ using the Hamiltonian $H^C + \mathbf{U} \cdot \Phi$ result the followings. Due to the degeneracy of the cross product, only two components of the Lagrange multipliers \mathbf{U} can be solved from these equations and a secondary constraint

$$\Phi = m\zeta \mathbf{X} \cdot \dot{\mathbf{X}} + \mathbf{X} \cdot \mathbf{P}^0 \quad (53)$$

arises. This secondary constraint Φ can also be derived directly by the dot product of momenta \mathbf{P}^0 with \mathbf{X} . We add the secondary constraint Φ to the Hamiltonian function and define the total Hamiltonian as

$$H_T^C = H^C + \mathbf{U} \cdot \Phi + U\Phi. \quad (54)$$

Using the total Hamiltonian H_T^C , the consistency conditions give the set of four equations

$$\dot{\Phi} = \{\Phi, H_T^C\} = \dot{\mathbf{X}} \cdot \mathbf{B} + \mathbf{U} \cdot \mathbf{A} \approx 0, \quad (55)$$

$$\dot{\Phi} = \{\Phi, H_T^C\} = -m\zeta \dot{\mathbf{X}} - \mathbf{P}^0 + \frac{\zeta^2}{m\mu} \mathbf{U} \times \mathbf{X} - U\mathbf{A} \approx 0. \quad (56)$$

Here, we used the abbreviations

$$\mathbf{A} = m\zeta \mathbf{X} + \mathbf{P}^1 \quad \text{and} \quad \mathbf{B} = m\zeta \dot{\mathbf{X}} + \mathbf{P}^0. \quad (57)$$

Under the assumption of $X^2 \neq 0$, we solve the Lagrange multipliers U and \mathbf{U} as follows

$$\begin{aligned} U &= \frac{-1}{m\zeta X^2} \mathbf{X} \cdot \mathbf{B} = \frac{-1}{m\zeta X^2} \Phi, \\ \mathbf{U} &= \frac{-3}{2m\zeta X^2} \mathbf{X} (\mathbf{B} \cdot \dot{\mathbf{X}}) - \frac{\mu m}{\zeta^2 X^2} (\mathbf{B} \times \mathbf{X}) \end{aligned} \quad (58)$$

where we used $\mathbf{X} \cdot \mathbf{P}^1 = 0$ from definition of momenta. Interesting to note that, U vanishes on the constraint submanifold, that is $U \approx 0$. Note that, since we start with Minkowskian metric, the condition $X^2 = 0$ refers that the particle is on the light cone. A naive way for relaxing the condition is to take the limit $X^2 \rightarrow 0$ in the expressions. Under this limit, the vector \mathbf{U} approximate the acceleration

$\ddot{\mathbf{X}}$ after the substitution of the momenta. This observation encourages us to comment on the continuous dependence of the Lagrange multipliers on the X^2 .

After substitution of the constraints Φ and Φ in (51 and 53), and the Lagrange multipliers U and \mathbf{U} in (58) into implicit form of H_T^C in (54), we write the total Hamiltonian as

$$\begin{aligned} H_T^C = & \frac{1}{2} \dot{\mathbf{X}} \cdot \mathbf{P}^0 - \frac{3}{2m\zeta\mathbf{X}^2} (\mathbf{X} \cdot \mathbf{P}^1) (\mathbf{B} \cdot \dot{\mathbf{X}}) - \frac{\mu m}{\zeta^2 \mathbf{X}^2} \mathbf{P}^1 \cdot (\mathbf{B} \times \mathbf{X}) \\ & + \frac{1}{2\mathbf{X}^2} (\mathbf{B} \cdot \mathbf{X}) (\mathbf{X} \cdot \dot{\mathbf{X}}) - \frac{1}{m\zeta\mathbf{X}^2} (\mathbf{B} \cdot \mathbf{X})^2 \end{aligned} \quad (59)$$

Using the total Hamiltonian H_T^C , we write the Hamilton's equations as

$$\begin{aligned} \dot{\mathbf{X}} & \approx \frac{1}{2} \dot{\mathbf{X}} + \frac{\mu m}{\zeta^2 \mathbf{X}^2} \mathbf{P}^1 \times \mathbf{X} + \frac{1}{2\mathbf{X}^2} \mathbf{X} (\mathbf{X} \cdot \dot{\mathbf{X}}) \\ \ddot{\mathbf{X}} & \approx \frac{\mu m}{\zeta^2 \mathbf{X}^2} \mathbf{X} \times \mathbf{B} - \frac{3}{2m\zeta\mathbf{X}^2} (\mathbf{B} \cdot \dot{\mathbf{X}}) \mathbf{X} \\ \dot{\mathbf{P}}^0 & \approx \frac{\mu m}{\zeta^2 \mathbf{X}^2} \mathbf{P}^1 \times \mathbf{B} + \frac{3}{2m\zeta\mathbf{X}^2} (\mathbf{B} \cdot \dot{\mathbf{X}}) \mathbf{P}^1 - \frac{1}{2\mathbf{X}^2} (\dot{\mathbf{X}} \cdot \mathbf{X}) \mathbf{B} \\ & \quad - \frac{2}{\mathbf{X}^4} \frac{\mu m}{\zeta^2} (\mathbf{P}^1 \cdot \mathbf{B} \times \mathbf{X}) \mathbf{X} \\ \dot{\mathbf{P}}^1 & \approx -\frac{1}{2} \mathbf{P}^0 - \frac{\mu m^2}{\zeta \mathbf{X}^2} \mathbf{P}^1 \times \mathbf{X} - \frac{m\zeta}{2\mathbf{X}^2} \mathbf{X}^i (\mathbf{X} \cdot \dot{\mathbf{X}}). \end{aligned} \quad (60)$$

Here, the first and the last of equations are identically satisfied. The Eq.(??) gives the weak equality $\ddot{\mathbf{X}} \approx \mathbf{U}$ and the third equation (??) gives

$$\dot{\mathbf{P}}^0 = \frac{\zeta^2}{2m\mu} \dot{\mathbf{X}} \times \ddot{\mathbf{X}} \quad (61)$$

which, using the definition of \mathbf{P}^0 , results in the Euler-Lagrange equations (44) for Clément Lagrangian.

4.3 Dirac-Poisson Bracket

All of the four constraints Φ and Φ are second class. This enables us to define the following nondegenerate 4×4 matrix

$$\mathcal{M} = \begin{pmatrix} \{\Phi, \Phi\} & \{\Phi, \Phi\} \\ \{\Phi, \Phi\} & \{\Phi, \Phi\} \end{pmatrix} = \begin{pmatrix} -\frac{\zeta^2}{\mu m} \hat{\mathbf{X}} & -m\zeta\mathbf{X} - \frac{\zeta^2}{2\mu m} \mathbf{X} \times \dot{\mathbf{X}} \\ m\zeta\mathbf{X}^T + \frac{\zeta^2}{\mu m} (\mathbf{X} \times \dot{\mathbf{X}})^T & 0 \end{pmatrix} \quad (62)$$

where super script T denoted the transpose of the vectors and the notation $\hat{\cdot}$ stands for the hatmap in (47). The determinant of the matrix \mathcal{M} is $\zeta^6 X^2 / \mu^2$ and its inverse is

$$\mathcal{M}^{-1} = \frac{1}{m\zeta X^2} \begin{pmatrix} \frac{1}{\zeta^2} m\mu (\hat{\mathbf{X}} + \hat{\mathbf{P}}^1) & \mathbf{X} \\ -\mathbf{X}^T & 0 \end{pmatrix}.$$

Recalling the definition (15) of the Dirac bracket, we construct

$$\begin{aligned}
\{X^i, \dot{X}^j\}_{DB} &= \frac{-1}{m\zeta X^2} X^i X^j, \\
\{\dot{X}^i, \dot{X}^j\}_{DB} &= \frac{-\mu\epsilon^{ijk}}{\zeta^3 X^2} A_k, \\
\{X^i, P_j^0\}_{DB} &= \delta_j^i - \frac{\zeta}{2\mu m^2 X^2} X^i \epsilon_{jkl} \dot{X}^k X^l, \\
\{\dot{X}^i, P_j^1\}_{DB} &= \delta_j^i - \frac{1}{2m\zeta X^2} (A_k X^k \delta_j^i - A_j X^i) - \frac{1}{X^2} X^i \delta_{jl} X^l, \\
\{\dot{X}^i, P_j^0\}_{DB} &= \frac{1}{2m\zeta X^2} (\delta_j^i \dot{X}^k A_k - \dot{X}^i A_j - 2X^i B_j), \\
\{P_i^0, P_j^0\}_{DB} &= \frac{\zeta}{4\mu m^2 X^2} (\epsilon_{ilj} \dot{X}^l \dot{X}^r A_r - 2\epsilon_{isn} \dot{X}^s X^n B_j + 2\epsilon_{jnr} \dot{X}^n X^r B_i), \\
\{P_i^0, P_j^1\}_{DB} &= \frac{\zeta}{4\mu m^2 X^2} (\epsilon_{ikl} \dot{X}^k X^l A_j - \epsilon_{jik} \dot{X}^k A_r X^r - 2m\zeta \epsilon_{rik} \dot{X}^k X^r \delta_{jl} X^l), \\
\{P_i^1, P_j^1\}_{DB} &= \frac{-\zeta}{4\mu m^2 X^2} \epsilon_{jki} X^k A_r X^r
\end{aligned} \tag{63}$$

where the rest is zero and the abbreviations defined in Eq.(57) are used. In terms of the Dirac brackets, the equations of motions are

$$\dot{\mathbf{X}} = \{\mathbf{X}, H^C\}_{DB} = \frac{-1}{m\zeta X^2} \mathbf{X}(\mathbf{X} \cdot \mathbf{B}) + \dot{\mathbf{X}} \tag{64}$$

$$\ddot{\mathbf{X}} = \{\dot{\mathbf{X}}, H^C\}_{DB} = \frac{-\mu}{\zeta^3 X^2} \mathbf{B} \times \mathbf{A} - \frac{1}{m\zeta X^2} \mathbf{X}(\mathbf{B} \cdot \dot{\mathbf{X}}) \tag{65}$$

$$\begin{aligned}
\dot{\mathbf{P}}^0 = \{\mathbf{P}^0, H^C\}_{DB} &= \frac{-1}{2m\zeta X^2} \mathbf{B}(\mathbf{A} \cdot \dot{\mathbf{X}}) + \frac{1}{2m\zeta X^2} \mathbf{A}(\mathbf{B} \cdot \dot{\mathbf{X}}) \\
&+ \frac{1}{m\zeta X^2} \mathbf{B}(\mathbf{X} \cdot \mathbf{B}) - \frac{\zeta}{2\mu m^2 X^2} \dot{\mathbf{X}} \times \mathbf{X}(\mathbf{B} \cdot \dot{\mathbf{X}})
\end{aligned} \tag{66}$$

$$\dot{\mathbf{P}}^1 = \{\mathbf{P}^1, H^C\}_{DB} = -\mathbf{B} + \frac{1}{2m\zeta X^2} \mathbf{B}(\mathbf{A} \cdot \mathbf{X}) - \frac{1}{2m\zeta X^2} \mathbf{A}(\mathbf{B} \cdot \mathbf{X}) + \frac{1}{X^2} \mathbf{X}(\mathbf{X} \cdot \mathbf{B}) \tag{67}$$

generated by the canonical Hamiltonian H^C in Eq.(50). Where we used the abbreviations $\mathbf{A} = m\zeta \mathbf{X} + \mathbf{P}^1$, $\mathbf{B} = m\zeta \dot{\mathbf{X}} + \mathbf{P}^0$ and $\mathbf{X}^2 = \mathbf{X} \cdot \mathbf{X}$. The first and fourth equations are identically satisfied. The third equation reduces to the Euler-Lagrange equations in the form of Eq.(61). The second equation gives

$$\mathbf{X} \cdot \ddot{\mathbf{X}} + \frac{3}{2\mu m^2 X^2} \dot{\mathbf{X}} \cdot (\ddot{\mathbf{X}} \times \mathbf{X}) = 0 \tag{68}$$

which is nothing but the dot product of the Euler-Lagrange Eq.(44) and \mathbf{X} , hence equal to zero modulo Euler-Lagrange equations.

4.4 Skinner-Rusk Unified Formalism

In order to put Clément dynamics in the form of Skinner-Rusk formalism, we recall Pontryagin bundle $P^3M = T^3M \times_{TM} T^*TM$ and the presymplectic two-form Ω_{P^3M} in Eq.(20). The Hamiltonian function on the presymplectic manifold (P^3M, Ω_{P^3M}) is

$$H_{P^3M} = \mathbf{P}^0 \cdot \dot{\mathbf{X}} + \mathbf{P}^1 \cdot \ddot{\mathbf{X}} + \frac{m\zeta}{2} \dot{X}^2 + \frac{2m\Lambda}{\zeta} - \frac{\zeta^2}{2\mu m} \mathbf{X} \cdot (\dot{\mathbf{X}} \times \ddot{\mathbf{X}})$$

and it generates the dynamics according to the Hamilton's equations $i_{X_{P^3M}} \Omega_{P^3M} = -dH_{P^3M}$. We recall the general form

$$\begin{aligned} X_{P^3M} &= \dot{\mathbf{X}} \cdot \nabla_{\mathbf{X}} + \ddot{\mathbf{X}} \cdot \nabla_{\dot{\mathbf{X}}} + \ddot{\mathbf{X}} \cdot \nabla_{\ddot{\mathbf{X}}} + \mathbf{C} \cdot \nabla_{\ddot{\mathbf{X}}} \\ &\quad + \frac{\zeta^2}{2\mu m} (\dot{\mathbf{X}} \times \ddot{\mathbf{X}}) \cdot \nabla_{\mathbf{P}^0} + \left(-m\zeta \dot{\mathbf{X}} - \frac{\zeta^2}{2\mu m} \mathbf{X} \times \ddot{\mathbf{X}} - \mathbf{P}^0 \right) \cdot \nabla_{\mathbf{P}^1} \end{aligned} \quad (69)$$

with unknown coefficient functions $\mathbf{C} = (C_1, C_2, C_3)$ which will be determined through the algorithm runs. The graph of the Legendre map \mathcal{FL} is defined by the constraints

$$\psi = \mathbf{P}^0 + m\zeta \dot{\mathbf{X}} + \frac{\zeta^2}{\mu m} \mathbf{X} \times \ddot{\mathbf{X}}, \quad \Phi = \mathbf{P}^1 - \frac{\zeta^2}{2\mu m} \mathbf{X} \times \dot{\mathbf{X}}. \quad (70)$$

The tangency conditions for the constraint functions are

$$\begin{aligned} X_{P^3M} \psi &= \psi_1 = \frac{3\zeta^2}{2\mu m} \dot{\mathbf{X}} \times \ddot{\mathbf{X}} + m\zeta \ddot{\mathbf{X}} + \frac{\zeta^2}{\mu m} \mathbf{X} \times \ddot{\mathbf{X}} \\ X_{P^3M} \Phi &= -\psi. \end{aligned} \quad (71)$$

$\psi_1 \approx 0$ defines the constraint manifold W_1 . We need to check tangency condition $X_{P^3M} \psi_1 \approx 0$ to decide that whether W_0 is the final constraint submanifold or not. Accordingly, we compute

$$X_{P^3M} \psi_1 = \frac{5\zeta^2}{2\mu m} \dot{\mathbf{X}} \times \ddot{\mathbf{X}} + m\zeta \ddot{\mathbf{X}} + \frac{\zeta^2}{\mu m} \mathbf{X} \times \mathbf{C}. \quad (72)$$

By requiring $X_{P^3M} \psi_1$ be zero, we obtain following information. First of all, the rank of the matrix $\hat{\mathbf{X}}$ is 2 (except from $\mathbf{X} = \mathbf{0}$) out of 3, hence solving \mathbf{C} uniquely from $X_{P^3M} \psi_1 = \mathbf{0}$ is not possible. We can only get two components of $\mathbf{C} = (C_1, C_2, C_3)$ in terms of the third one, say we solve C_1 and C_2 in terms of C_3 . In addition, from Eq.(72), we get a single secondary constraint

$$\psi_2 = m\zeta \mathbf{X} \cdot \ddot{\mathbf{X}} + \frac{5\zeta^2}{2\mu m} \mathbf{X} \cdot \dot{\mathbf{X}} \times \ddot{\mathbf{X}}$$

which is obtained by taking dot product of Eq.(72) with \mathbf{X} . Hence, we arrive at the submanifold W_2 which is the final constraint submanifold since the requirement that

$$X_{P^3M} \psi_2 = \ddot{\mathbf{X}} \cdot \left(m\zeta \dot{\mathbf{X}} + \frac{5\zeta^2}{2\mu m} \mathbf{X} \times \ddot{\mathbf{X}} \right) + \mathbf{C} \cdot \left(m\zeta \mathbf{X} + \frac{5\zeta^2}{2\mu m} \mathbf{X} \times \dot{\mathbf{X}} \right) = 0 \quad (73)$$

leads us to determine C_3 and substitute in C_1 and C_2 . Thus,

$$\begin{aligned} \mathbf{C} = & \frac{-1}{m\zeta X^2} \mathbf{X} (\ddot{\mathbf{X}} \cdot (m\zeta \dot{\mathbf{X}} + \frac{5\zeta^2}{2\mu m} \mathbf{X} \times \dot{\mathbf{X}})) \\ & + \frac{\mu}{\zeta^3 X^2} (m\zeta \mathbf{X} + \frac{5\zeta^2}{2\mu m} \mathbf{X} \times \dot{\mathbf{X}}) \times (m\zeta \ddot{\mathbf{X}} + \frac{5\zeta^2}{2\mu m} \ddot{\mathbf{X}} \times \dot{\mathbf{X}}). \end{aligned} \quad (74)$$

Note that, the substitutions of Legendre map in H_{P^3M} leads to the Lagrangian energy

$$E_{T^3M} = -\frac{m\zeta}{2} \dot{X}^2 + \frac{\zeta^2}{\mu m} \mathbf{X} \cdot (\dot{\mathbf{X}} \times \ddot{\mathbf{X}}) + \frac{2m\Lambda}{\zeta} \quad (75)$$

on T^3M . On the projection $S_f = pr_1(W_f)$ of W_f in (??), the Hamilton's equations are given by means of the presymplectic relation

$$i_{X_{T^3M}} \Omega_{T^3M} + dE_{T^3M}|_{S_f} = 0, \quad (76)$$

where Ω_{T^3M} is the presymplectic structure on T^3M in Eq.(??). Here,

$$X_{T^3M} = (pr_1)_* X_{P^3M} = \dot{\mathbf{X}} \cdot \nabla_{\mathbf{X}} + \ddot{\mathbf{X}} \cdot \nabla_{\dot{\mathbf{X}}} + \ddot{\mathbf{X}} \cdot \nabla_{\ddot{\mathbf{X}}} + \mathbf{C} \cdot \nabla_{\ddot{\mathbf{X}}}. \quad (77)$$

From presymplectic relation (76), we can derive the Euler-Lagrange equations (44).

5 Conclusions

In the present work, we have derived the Hamiltonian and the Skinner-Rusk unified formalisms for the second order degenerate Sarioğlu-Tekin and Clément Lagrangians. We have applied the Dirac-Bergmann constraint algorithm in order to arrive at the Hamiltonian pictures on the momentum phase spaces whereas we have applied the Gotay-Nester-Hinds algorithm while investigating the Skinner-Rusk unified formalisms on the proper Whitney bundles. As a result, for the Sarioğlu-Tekin Lagrangian (2), we have obtained the total Hamiltonian function in (32), the Hamilton's equations in Eq.(33-34), the Dirac-Poisson bracket in Eqs.(36), and the vector field generating the unified formalism in Eq.(40). For the Clément Lagrangian, we have calculated the constraint Hamiltonian function in Eq.(59), the Hamilton's equations in Eqs.(60), the Dirac-Poisson bracket in Eqs.(63), and the vector field generating the unified formalism in Eq.(77).

Here is the list of some possible complementary and future works:

- We are planning to study the Hamiltonian formalism of Sarioğlu-Tekin and Clément Lagrangian after writing them as first order Lagrangians by properly defining new coordinates and Lagrange multipliers. By this, we can able to make a comparative study of the Hamiltonian representations of the degenerate second order Lagrangians and their reduced degenerate first order equivalents on some concrete problems.

- Schmidt-Legendre transformation is an alternative method while reducing a second order Lagrangian function to the first order. It works both for non-degenerate and degenerate systems [64, 65]. In this theory, the acceleration is assumed as a new coordinate instead of the velocity [2, 3, 27]. We are planning to make a detail analysis of the present discussions in terms of the Schmidt-Legendre transformation.
- Both of the Sarioğlu-Tekin and Clément Lagrangians have rotational symmetry. In [28], the higher dimensional version of the Lagrangian reduction [10, 47] has been presented. By motivating this, we are planning to exhibit formal reductions of the Sarioğlu-Tekin and Clément Lagrangians under rotational symmetry.

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